Pattern matching without K

Jesper Cockx
Dominique Devriese
Frank Piessens

DistriNet – KU Leuven

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How can we recognize definitions by pattern matching that do not depend on K?

By taking identity proofs into account during unification of the indices!

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By taking identity proofs into account during unification of the indices!

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Simple pattern matching

```
data ℕ : Set where
  z : ℕ
  s : ℕ → ℕ

min : ℕ → ℕ → ℕ
min   x     y  =  ?
```

Simple pattern matching

data $\mathbb{N}$: `Set` **where**
  `z` : $\mathbb{N}$
  `s` : $\mathbb{N} \to \mathbb{N}$

`min` : $\mathbb{N} \to \mathbb{N} \to \mathbb{N}$
`min` `z` y = `z`
`min` (`s` x) y = ?

Simple pattern matching

```
data ℕ : Set where
  z : ℕ
  s : ℕ → ℕ

min : ℕ → ℕ → ℕ
min z y = z
min (s x) z = z
min (s x) (s y) = s (min x y)
```

Dependent pattern matching

$$\textbf{data}\ \_\leq\_ : \mathbb{N} \to \mathbb{N} \to \mathtt{Set}\ \textbf{where}$$
$$\quad \mathtt{lz} : (x : \mathbb{N}) \to \mathtt{z} \leq x$$
$$\quad \mathtt{ls} : (x\ y : \mathbb{N}) \to x \leq y \to \mathtt{s}\ x \leq \mathtt{s}\ y$$

$$\mathtt{antisym} : (x\ y : \mathbb{N}) \to x \leq y \to y \leq x \to x \equiv y$$
$$\mathtt{antisym}\quad x \quad y \quad p \quad q \quad = \quad ?$$

Dependent pattern matching

```
data _≤_ : ℕ → ℕ → Set where
  lz : (x : ℕ) → z ≤ x
  ls : (x y : ℕ) → x ≤ y → s x ≤ s y

antisym : (x y : ℕ) → x ≤ y →        y ≤ x  →  x ≡ y
antisym   ⌊z⌋   ⌊y⌋   (lz y)         q       =   ?
antisym   ⌊s x⌋ ⌊s y⌋ (ls x y p)     q       =   ?
```

Dependent pattern matching

$$\textbf{data}\ \_\leq\_ : \mathbb{N} \to \mathbb{N} \to \texttt{Set}\ \textbf{where}$$
$$\quad \texttt{lz} : (x : \mathbb{N}) \to \texttt{z} \leq x$$
$$\quad \texttt{ls} : (x\ y : \mathbb{N}) \to x \leq y \to \texttt{s}\ x \leq \texttt{s}\ y$$

$$\begin{array}{llllll}
\texttt{antisym} : (x\ y : \mathbb{N}) \to & x \leq y \to & y \leq x & \to & x \equiv y \\
\texttt{antisym}\ \lfloor \texttt{z} \rfloor\ \lfloor \texttt{z} \rfloor & (\texttt{lz}\ \lfloor \texttt{z} \rfloor) & (\texttt{lz}\ \lfloor \texttt{z} \rfloor) & = & \texttt{refl} \\
\texttt{antisym}\ \lfloor \texttt{s}\ x \rfloor\ \lfloor \texttt{s}\ y \rfloor & (\texttt{ls}\ x\ y\ p) & q & = & ?
\end{array}$$

Dependent pattern matching

```
data _≤_ : ℕ → ℕ → Set where
  lz : (x : ℕ) → z ≤ x
  ls : (x y : ℕ) → x ≤ y → s x ≤ s y

antisym : (x y : ℕ) → x ≤ y →   y ≤ x → x ≡ y
antisym  ⌊z⌋   ⌊z⌋   (lz ⌊z⌋)   (lz ⌊z⌋) = refl
antisym  ⌊s x⌋ ⌊s y⌋ (ls x y p) (ls ⌊y⌋ ⌊x⌋ q)
    = cong s (antisym x y p q)
```

$$
\begin{array}{l}
\mathtt{antisym} : (m\ n : \mathbb{N}) \to m \leq n \to n \leq m \to m \equiv n \\
\mathtt{antisym} = \mathtt{elim}_{\leq}\ (\lambda m; n; \_.\ n \leq m \to m \equiv n) \\
\quad (\lambda n; e.\ \mathtt{elim}_{\leq}\ (\lambda n; m; \_.\ m \equiv \mathtt{z} \to m \equiv n) \\
\qquad (\lambda n; e.\ e) \\
\qquad (\lambda k; l; \_; \_; e.\ \mathtt{elim}_{\bot}(\lambda \_.\ \mathtt{s}\ l \equiv \mathtt{s}\ k) \\
\qquad\quad (\mathtt{noConf}_{\mathbb{N}}\ (\mathtt{s}\ l)\ \mathtt{z}\ e)) \\
\qquad n\ \mathtt{z}\ e\ \mathtt{refl}) \\
\quad (\lambda m; n; \_; H; q.\ \mathtt{cong}\ \mathtt{s} \\
\qquad (H \\
\qquad\quad (\mathtt{elim}_{\leq}\ (\lambda k; l; \_.\ k \equiv \mathtt{s}\ n \to l \equiv \mathtt{s}\ m \to n \leq m) \\
\qquad\qquad (\lambda \_; e; \_.\ \mathtt{elim}_{\bot}\ (\lambda \_.\ n \leq m) \\
\qquad\qquad\quad (\mathtt{noConf}_{\mathbb{N}}\ \mathtt{z}\ (\mathtt{s}\ n)\ e)) \\
\qquad\qquad (\lambda k; l; e; \_; p; q.\ \mathtt{subst}\ (\lambda n.\ n \leq m) \\
\qquad\qquad\quad (\mathtt{noConf}_{\mathbb{N}}\ (\mathtt{s}\ k)\ (\mathtt{s}\ n)\ p) \\
\qquad\qquad\quad (\mathtt{subst}\ (\lambda m.\ k \leq m) \\
\qquad\qquad\qquad (\mathtt{noConf}_{\mathbb{N}}\ (\mathtt{s}\ l)\ (\mathtt{s}\ m)\ q)\ e)) \\
\qquad\qquad (\mathtt{s}\ n)\ (\mathtt{s}\ m)\ q\ \mathtt{refl}\ \mathtt{refl})))
\end{array}
$$

The identity type as an inductive family

```
data _≡_ (x : A) : A → Set where
  refl : x ≡ x

trans : (x y z : A) → x ≡ y → y ≡ z → x ≡ z
trans x ⌊x⌋ ⌊x⌋ refl refl = refl
```

The identity type as an inductive family

$$\textbf{data}\ \_\equiv\_\ (x : A) : A \to \texttt{Set}\ \textbf{where}$$
$$\quad \texttt{refl} : x \equiv x$$

$$\texttt{trans} : (x\ y\ z : A) \to x \equiv y \to y \equiv z \to x \equiv z$$
$$\texttt{trans}\ x\ \lfloor x \rfloor\ \lfloor x \rfloor\ \texttt{refl}\ \texttt{refl} = \texttt{refl}$$

K follows from pattern matching

$$\begin{aligned}
&\mathtt{K} : (P : a \equiv a \to \mathtt{Set}) \to \\
&\qquad (p : P\ \mathtt{refl}) \to \\
&\qquad (e : a \equiv a) \to P\ e \\
&\mathtt{K}\quad P\ p\ \mathtt{refl} = p
\end{aligned}$$

We don't always want to assume K

K is incompatible with univalence:

- K implies that $\texttt{subst}\ e\ \texttt{true} = \texttt{true}$ for all $e : \texttt{Bool} \equiv \texttt{Bool}$
- Univalence gives $\texttt{swap} : \texttt{Bool} \equiv \texttt{Bool}$ such that $\texttt{subst swap true} = \texttt{false}$

hence $\texttt{true} = \texttt{false}$!

Pattern matching without K

Unification of the indices

$$x \simeq x, \Delta \Rightarrow \Delta \qquad \text{(Deletion)}$$

$$t \simeq x, \Delta \Rightarrow \Delta[x \mapsto t] \qquad \text{(Solution)}$$

$$\mathsf{c}\ \bar{s} \simeq \mathsf{c}\ \bar{t}, \Delta \Rightarrow \bar{s} \simeq \bar{t}, \Delta \qquad \text{(Injectivity)}$$

$$\mathsf{c}_1\ \bar{s} \simeq \mathsf{c}_2\ \bar{t}, \Delta \Rightarrow \bot \qquad \text{(Conflict)}$$

$$x \simeq \mathsf{c}\ \bar{p}[x], \Delta \Rightarrow \bot \qquad \text{(Cycle)}$$

The criterium

- It is not allowed to delete reflexive equations.
- When applying injectivity on an equation $\texttt{c}\ \bar{s} = \texttt{c}\ \bar{t}$ of type $\texttt{D}\ \bar{u}$, the indices $\bar{u}$ should be *self-unifiable*.

Why deletion has to be disabled

$$\texttt{UIP} : (e : a \equiv a) \rightarrow e \equiv \texttt{refl}$$
$$\texttt{UIP}\ \ \underline{\texttt{refl}} = \texttt{refl}$$

Couldn't solve reflexive equation $a = a$ of type A because K has been disabled.

Why injectivity has to be restricted

$$\texttt{UIP}' : (e : \texttt{refl} \equiv_{a \equiv a} \texttt{refl}) \rightarrow e \equiv \texttt{refl}$$
$$\texttt{UIP}' \quad \underline{\texttt{refl}} = \texttt{refl}$$

Couldn't solve reflexive equation $a = a$ of type A because K has been disabled.

Pattern matching without K

Eliminating dependent pattern matching

1. Basic case analysis:
   Translate each case split to an eliminator.
2. Specialization by unification:
   Solve the equations on the indices.
3. Structural recursion:
   Fill in the recursive calls.

Heterogeneous equality

$$\frac{a : A \qquad b : B}{a \simeq b : \texttt{Set}} \qquad\qquad \frac{a : A}{\texttt{refl} : a \simeq a}$$

$$\texttt{eqElim} : (x\ y : A) \to (e : x \simeq y) \to \\ D\ x\ \texttt{refl} \to D\ y\ e$$

This elimination rule is equivalent with K …

Homogeneous telescopic equality

We can use the first equality proof
to fix the types of the following equations.

$$a_1, a_2 \equiv b_1, b_2$$
$$\Downarrow$$
$$(e_1 : a_1 \equiv b_1)(e_2 : \texttt{subst}\ e_1\ a_2 \equiv b_2)$$

Deletion

$$
\begin{array}{c}
x \simeq x, \Delta \Rightarrow \Delta \\
\Downarrow \\
\mathtt{e} : x \equiv x, \Delta \Rightarrow \Delta[\mathtt{e} \mapsto \mathtt{refl}]
\end{array}
$$

This is exactly the K axiom!

Solution

$$t \simeq x, \Delta \Rightarrow \Delta[x \mapsto t]$$

$$\Downarrow$$

$$\mathtt{e} : t \equiv x, \Delta \Rightarrow \Delta[x \mapsto t, \mathtt{e} \mapsto \mathtt{refl}]$$

Injectivity

$$
\begin{array}{ccc}
\mathsf{c}\ \bar{s} \simeq \mathsf{c}\ \bar{t}, \Delta & \Rightarrow & \bar{s} \simeq \bar{t}, \Delta \\
& \Downarrow & \\
\mathsf{e} : \mathsf{c}\ \bar{s} \equiv \mathsf{c}\ \bar{t}, \Delta & \Rightarrow & \bar{\mathsf{e}} : \bar{s} \equiv \bar{t}, \Delta[\mathsf{e} \mapsto \mathtt{conf}\ \bar{\mathsf{e}}]
\end{array}
$$

Indices of $\mathsf{c}\ \bar{s}$ and $\mathsf{c}\ \bar{t}$ should be unifiable

Conflict

$$\mathsf{c}_1\ \bar{u} \simeq \mathsf{c}_2\ \bar{v}, \Delta \Rightarrow \bot$$
$$\Downarrow$$
$$\mathsf{e} : \mathsf{c}_1\ \bar{s} \equiv \mathsf{c}_2\ \bar{t}, \Delta \Rightarrow \bot$$

Cycle

$$x \simeq \mathsf{c}\ \bar{p}[x], \Delta \Rightarrow \bot$$

$$\Downarrow$$

$$\mathsf{e} : x \equiv \mathsf{c}\ \bar{p}[x], \Delta \Rightarrow \bot$$

Possible extensions

- Detecting types that satisfy K (i.e. sets)
- Implementing the translation to eliminators
- Extending pattern matching
  to higher inductive types

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Conclusion

By restricting the unification algorithm,
we can make sure that K is never used.

You no longer have to worry
when using pattern matching for HoTT!

http://people.cs.kuleuven.be/
~ jesper.cockx/Without-K/

Standard library without K
Fixable errors: 16

Module	Functions
Algebra.RingSolver	$\stackrel{?}{=}$H, $\stackrel{?}{=}$N
Data.Fin.Properties	drop-suc
Data.Vec.Equality	trans, $\stackrel{?}{=}$
Data.Vec.Properties	::-injective, ...
Relation.Binary.Vec.Pointwise	head, tail
Data.Fin.Subset.Properties	drop-there, $\notin\bot$, ...
Data.Fin.Dec	$\in$?
Data.List.Countdown	drop-suc

Unfixable/unknown errors: 20

Module	Functions
Relation.Binary.	
HeterogeneousEquality	$\cong$-to-$\equiv$, subst, cong, …
PropositionalEquality	proof-irrelevance
Sigma.Pointwise	Rel$\leftrightarrow\equiv$, inverse
Data.	
Colist	Any-cong, $\sqsubseteq$-Poset
Covec	setoid
Container.Indexed	setoid, natural, $\circ$-correct
List.Any.BagAndSetEquality	drop-cons
Star.Decoration	gmapAll, $\triangleleft\triangleleft\triangleleft$
Star.Pointer	lookup
Vec.Properties	proof-irrelevance-[]=