Connecting Agda to other theorem provers via EuroProofNet

or, how to implement an Agda backend

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From Agda to Dedukti

1. EuroProofNet and Dedukti
2. Encoding Agda in Dedukti
3. Translating Agda to Dedukti
4. Encoding universe polymorphism and eta equality
5. Conclusion

What is EuroProofNet?[1]

- A European network of researchers working on improving interoperability between proof systems.
- 300 researchers from 40 different countries (you're welcome to join too!).
- Centered around Dedukti as an intermediate language for translation.

[1] `https://europroofnet.github.io/`

What is Dedukti?[2]

- A logical framework based on the $\lambda\Pi$-calculus with higher-order rewrite rules.
- Used for checking and transforming proofs, as well as translating between proof systems.
- Has a small core, so it should be easy to implement many different independent checkers (currently 3).
- Has encodings of many proof assistants: Coq, Isabelle/HOL, Matita, ...
- Also has encodings of other proof systems: Zenon Modulo, iProver, veriT, ...

[2] `https://deducteam.github.io/`

One-slide Dedukti tutorial

```
ty  : Type.
arr : ty -> ty -> ty.
```

One-slide Dedukti tutorial

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arr : ty -> ty -> ty.

tm  : ty -> Type.
lam : a:ty -> b:ty
   -> (tm a -> tm b) -> tm (arr a b).
def id : a:ty -> tm (arr a a)
  := a => lam a a (x => x).
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def id : a:ty -> tm (arr a a)
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def app : a:ty -> b:ty
        -> tm (arr a b) -> tm a -> tm b.
[u,v] app _ _ (lam _ _ u) v --> u v.
def twice : a:ty -> tm (arr (arr a a) (arr a a))
  := a => lam (arr a a) (arr a a) (f =>
          lam a a (x => app a a f (app a a f x))).
```

A success story

Recently, Thiago Felicissimo and Frédéric Blanqui (CSL '23)[3] translated a big mathematical library from Matita to Agda via Dedukti.

This includes proofs of **Bertrand's Postulate** and **Fermat's Little Theorem**, which were so far not done in Agda.

Hence translating between proof assistants can *already save us work today*!

[3] `http://files.inria.fr/blanqui/pred.pdf`

Agda2Dedukti

To connect Agda to EuroProofNet, we need to:

- Encode basic logic of Agda in Dedukti
- Translate Agda files to this encoding in Dedukti

This is the goal of **Agda2Dedukti**.

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Encoding Agda's universe levels

```
Lvl : Type.
0   : Lvl.
S   : Lvl -> Lvl.

def max : Lvl -> Lvl -> Lvl.
[n]   max 0     n     --> n.
[m]   max m     0     --> m.
[m,n] max (S m) (S n) --> S (max m n).
```

(Universe polymorphism comes later.)

Tarski- vs. Russell-style universes[4]

Agda uses Russell-style universes:
Elements of a universe are types themselves.

$$\frac{A : \mathsf{U}_l}{A \ \ \text{TYPE}}$$

Dedukti uses a form of Tarski-style universes:
Elements are codes that can be *interpreted* as types.

$$\frac{A : \mathsf{U}_l}{\texttt{El}\ A \ \ \text{TYPE}}$$

$\Rightarrow$ Encoding Agda in Dedukti takes some overhead.

[4] `https://www.cs.rhul.ac.uk/home/zhaohui/universes.pdf`

Encoding Agda's sorts

We define Agda's sort system[5] in three easy steps:

1. Define a Dedukti type of Agda sorts, and add a concrete sort `set` *l* for each level *l*:

```
Sort : Type.
set  : Lvl -> Sort.
```

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U : s : Sort -> Type.
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   ```
   U : s : Sort -> Type.
   ```

3. Interpret each type *a* : `U` *s* as a Dedukti type `El` *s* *a*:

   ```
   def El : s : Sort -> a : U s -> Type.
   ```

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1. For each sort s, define its successor `axiom` s:

```
def axiom : Sort -> Sort.
[i] axiom (set i) --> set (S i).
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```

2. Add a constant `u` internalizing any sort *s* as a term `u` *s* of sort `axiom` *s*:

```
u : s : Sort -> U (axiom s).
```

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   ```
   u : s : Sort -> U (axiom s).
   ```

3. Identify elements of `u` s with the ones of `U` s:

   ```
   [s] El _ (u s) --> U s.
   ```

Encoding Agda's function types

Function types follow the same pattern:

1. For two types $A : s_1$ and $x : A \vdash B : s_2$, define the sort `rule` s_1 s_2 of the pi type $(x : A) \to B$:

```
def rule : Sort -> Sort -> Sort.
[i, j] rule (set i) (set j) --> set (max i j).
```

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2. Add a constant `prod` for encoding the pi type:

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prod : s1 : Sort -> s2 : Sort ->
  A : U s1 -> B : (El s1 A -> U s2) ->
  U (rule s1 s2).
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```

3. Identify elements of `prod` with Dedukti's arrow type:

```
[s1, s2, A, B] El _ (prod s1 s2 A B)
            --> x : El s1 A -> El s2 (B x).
```

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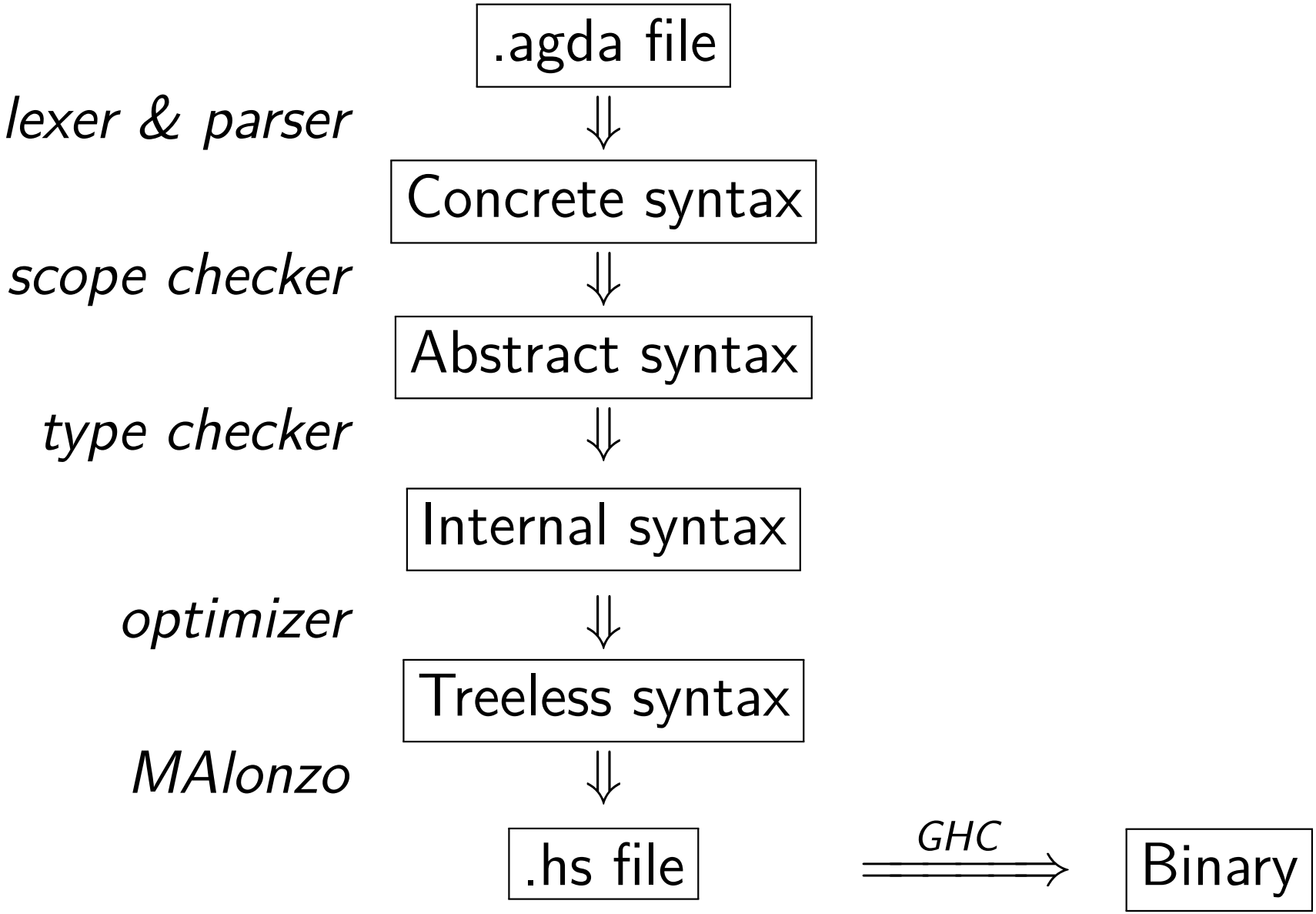
Implementation of Agda2Dedukti

Agda2Dedukti is implemented as an Agda backend.

This allows us to reuse parts of Agda's implementation:

- Internal syntax representation
- Type checking monad `TCM`

Structure of the Agda typechecker



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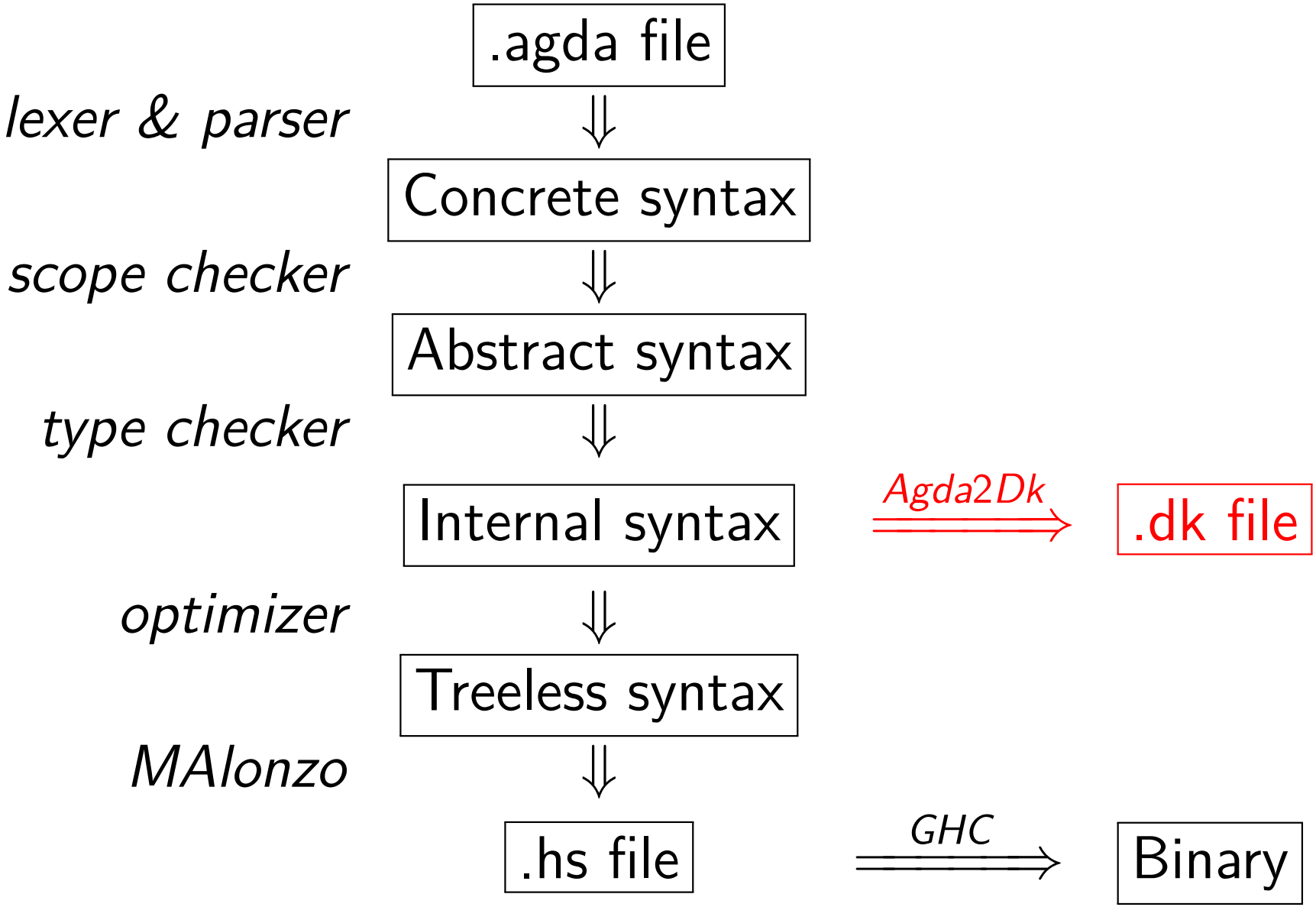

Agda's internal syntax[6]

```
data Term
  = Var Int Elims                        -- x u v ..
  | Lam ArgInfo (Abs Term)               -- λ x → v
  | Lit Literal                          -- 42, 'a', ...
  | Def QName Elims                      -- f u v ..
  | Con ConHead ConInfo Elims            -- c u v ..
  | Pi (Dom Type) (Abs Type)             -- (x : A) → B
  | Sort Sort                            -- Set, Set₁, Prop, ...
  | Level Level                          -- lzero, ...
  | MetaV MetaId Elims                   -- _X_235
  | DontCare Term
  | Dummy String Elims
```

[6] Code from `Agda.Syntax.Internal`

Agda's TCM monad

Agda's typechecker uses a type-checking monad TCM:

```
type TCM a
getConstInfo :: QName -> TCM Definition
getBuiltin :: String -> TCM Term
getContext :: TCM Context
addContext :: (Name, Dom Type) -> TCM a -> TCM a
checkInternal :: Term -> Type -> TCM ()
reconstructParameters :: Type -> Term -> TCM Term
...
```

Translating terms

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	$\mathtt{f}$
Constructor	$[\![\mathsf{c}]\!]$	$=$	$\mathtt{c}$
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	$\mathtt{prod}\ [\![\mathtt{sortOf}(A)]\!]_{\mathtt{s}}$ $[\![\mathtt{sortOf}(B)]\!]_{\mathtt{s}}$ $[\![A]\!]\ (x \Rightarrow [\![B]\!])$
Universe	$[\![s]\!]$	$=$	$\mathtt{u}\ [\![s]\!]_{\mathtt{s}}$
Sort	$[\![\mathsf{Set}\ \ell]\!]_{\mathtt{s}}$	$=$	$\mathtt{set}\ [\![\ell]\!]$

Translating datatypes and constructors

Data types and their constructors do not reduce, so we translate them to constants in Dedukti.

Example. Nat is translated to:

```
Nat  : U (set 0).
zero : El (set 0) Nat.
suc  : El (set 0)
  (prod (set 0) (set 0) Nat (_ => Nat)).
```

Translating clauses to rewrite rules

Functions in Agda are defined by a set of clauses, so we translate them to a constant + a set of rewrite rules. **Example.** pred is translated to:

```
def pred : El (set O)
  (prod (set O) (set O) Nat (_ => Nat)).
[]  pred zero     --> zero.
[m] pred (suc m) --> m.
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$\Rightarrow$ We extend the theory with each definition.

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Ongoing work: Instead, we can translate definitions by pattern matching to eliminators.[7]

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Levels are given by the syntax:

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Levels are not freely generated, they satisfy:

Idempotence: $a \sqcup a = a$

Associativity: $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$

Commutativity: $a \sqcup b = b \sqcup a$

Distributivity: $\mathsf{lsuc}\ (a \sqcup b) = \mathsf{lsuc}\ a \sqcup \mathsf{lsuc}\ b$

Neutrality: $a \sqcup \mathsf{lzero} = a$

Subsumption: $a \sqcup \mathsf{lsuc}^n\ a = \mathsf{lsuc}^n\ a$

Current solution: levels as sets

Idea. Every level l admits a unique canonical form

$$l = \max\{n, i_1 + m_1, ..., i_k + m_k\}$$

where $i_1, .., i_k \in FV(l)$, $n, m_1, .., m_k \in \mathbb{N}$ and $m_j \leq n$.

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But idempotence and subsumption require a non-linear rule:

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This *breaks confluence of pre-terms*, and prevents proving conservativity.

Eta equality for records

1. Eta for Σ:

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2. Eta for $\top$:

$$\frac{u : \top}{u = \mathsf{tt} : \top}$$

$\Rightarrow$ To check if two terms are convertible, it does not suffice to compare their normal forms.

Encoding eta

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   This is not stable under substitution:

$$(\lambda a : A.a)\{\mathbb{N} \times \mathbb{N}/A\}$$

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2. Eta-reduce everything when translating?
   This does not work for unit type, and needs non-linearity for the others:

   ```
   mk_pair (pi_1 p) (pi_2 p) --> p
   ```

Encoding eta

3. Annotate terms with their types to be able to match them to eta expand? e.g.

```
eta (arrow nat nat) f --> x => f x
```

[8] A. Bauer, A. Petković, An extensible equality checking algorithm for dependent type theories

Encoding eta

3. Annotate terms with their types to be able to match them to eta expand? e.g.

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   We get bigger terms, and the other rules make the system non-confluent on pre-terms.
   Moreover, variables not translated as variables.

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4. **The next idea.** Extend Dedukti with typed-directed rewrite rules, taking inspiration from Andromeda 2's extensionality rules.[8]

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Summary

Many parts of Agda can be encoded directly:

- Defined symbols $\mapsto$ constants
- Clauses $\mapsto$ rewrite rules

Other features require some more work:

- Reconstruction of constructor parameters
- Equational theory for universe levels

Yet other features we don't yet know how to encode (e.g. eta-equality for records).

Future work

- Compilation of clauses to elimination principles
- A conservative encoding of universe polymorphism
- An adequate and computational encoding of Agda[9]
- An encoding of eta-equality and irrelevance[10]

[9] For details, see Thiago's talk about Adequate and Computational Encodings in Dedukti, at FSCD 2022

[10] This probably requires extending Dedukti with type-aware conversion.

References

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[13] http://files.inria.fr/blanqui/pred.pdf